$$f_i(u_j) = (18\alpha\delta_{ik} - 9\beta\delta_{ik} + 2\,\delta_{ik})u_k^{n-2}/11 \qquad (46)$$

In view of Eq. (7) for each time increment, the Newton-Raphson iterative steps are governed by

$$u_j^{n+1,r+1} = u_j^{n+1,r} + (J_{ij}^{n+1,r})^{-1} R_j(u^{n+1,r}) \qquad (47)$$

where

$$J_{ij}^{n+1} = \partial[R_i^{n+1}(u_j^r)] \leftrightarrow / \partial u_j \qquad (48)$$

As given by Eq. (10), the convergence criterion is

$$(\partial J_{jm}^{n+1,r}/\partial u_m^r R_j^{n+1}(u_j^r) \leq J_{ik}^{n+1,r}\, J_{jk}^{n+1,r}\, \delta_{ij} \qquad (49)$$

The rate of convergence is evaluated by noting the error at the $r+1$ th step as

$$r_i^{r+1} = \tfrac{1}{2} e_i^r\, e_j^r [J_{jk}^{n+1,r}(u_i)]^{-1} (\partial J_{k\ell}^r / \partial u_\ell) \qquad (50)$$

It is seen that the Jacobian J_{ij}^{n+1}, which is the determining factor for convergence of the equations of nonlinear character, is affected by Δt since $R^{n+1}(u_j^r)$ is a function of Δt present in Eq. (44) or (45). It is clear, that the nonlinear time dependent finite element equations cannot be assured, unless Eqs. (38b) and (49) are simultaneously satisfied.

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Prediction of Recovery Factor and Reynolds' Analogy for Compressible Turbulent Flow

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Introduction

RECENTLY the so-called surface renewal and penetration model of turbulent transport in the vicinity of a wall has been applied to a wide variety of flow problems.[1-6] Of particular interest to workers in the field of high-speed flows are the prediction of recovery factor $[R = (T_{aw} - T_\infty)/(T_{0\infty} - T_\infty)]$, and the effects of viscous dissipation on the Reynolds' analogy factor $(RAF = 2St/C_f)$ in turbulent flows.

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In two recent papers, Thomas and Chung[5,6] have successfully applied the surface renewal model to the prediction of recovery factor and Reynolds' analogy factor including the effects of viscous dissipation; however, these analyses considered only the case of constant fluid properties, and hence their application to high-speed flows is not immediately obvious. It is the purpose of this Note to formulate the surface renewal and penetration model for compressible flows including viscous dissipation and to establish the approximate validity of the previous analyses for compressible gas flows.[5,6]

Analysis

The surface renewal and penetration model as first set forth by Danckwerts[7] is based on the assumption that macroscopic chunks of fluid ("eddies") intermittently move from the turbulent core into the close vicinity of the transport surface. During the time of residence in the wall region, unsteady one-dimensional molecular transport of momentum and energy are assumed to dominate. Several experimental investigations of incompressible turbulent flows are in basic agreement with this model,[8-10] except that the fluid elements do not move into direct contact with the wall; however, the assumption that they do reach the wall has been found to yield agreeable results for Prandtl numbers less than 10.[11]

Consider an eddy which moves from the turbulent core into contact with the wall. Neglecting axial gradients as small compared with stationary unsteady terms and transverse gradients, the continuity, momentum, and energy equations are

$$\frac{\partial \rho}{\partial \theta} + \frac{\partial(\rho v)}{\partial y} = 0 \qquad (1)$$

$$\rho\left(\frac{\partial u}{\partial \theta} + v\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \qquad (2)$$

$$\rho\left(\frac{\partial h}{\partial \theta} + v\frac{\partial h}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \dagger \qquad (3)$$

wherein θ is the instantaneous contact time, and a perfect gas $(h = C_pT)$ has been assumed. In Eqs. (2) and (3) the axial pressure gradient has been neglected. While nominally restricting the resulting analysis to flat plate flows, the pressure gradient term has, for incompressible flows, been shown to be negligible for tube flows above Reynolds numbers of 10^4(Ref. 2) and for boundary layers in mild pressure gradients.[3]

For flows not too near separation, the residence time of a typical eddy in the wall region is such that the eddy may be considered semi-infinite in the transverse direction.[3,4] It is also assumed that transfer of momentum and energy to the eddy during its flight from the turbulent core to the wall is negligible, thus the initial and boundary conditions on Eqs. (1-3) become

$$u = 0, \quad h = h_w \quad \text{at } y = 0 \qquad (4a)$$

$$u = U_\infty, \quad h = h_\infty \quad \text{at } y \to \infty \qquad (4b)$$

$$u = U_\infty, \quad h = h_\infty \quad \text{at } \theta = 0 \qquad (4c)$$

Equations (1-4) are now transformed via the introduction of

$$dY = (\rho/\rho_\infty)dy \qquad (5)$$

resulting in

$$\frac{\partial u}{\partial \theta} = \nu_\infty \frac{\partial}{\partial Y}\left[C\frac{\partial u}{\partial Y}\right] \qquad (6)$$

$$\frac{\partial T}{\partial \theta} = \frac{\nu_\infty}{Pr}\frac{\partial}{\partial Y}\left[C\frac{\partial T}{\partial Y}\right] + \frac{C\nu_\infty}{C_p}\left[\frac{\partial u}{\partial Y}\right]^2 \qquad (7)$$

†Although the transverse convection terms may be neglected, the transformation to be introduced causes them to drop out identically.

$$u=0, \quad T=T_w \quad \text{at } Y=0 \tag{8a}$$

$$u=U_\infty, \quad T=T_\infty \quad \text{at } Y\to\infty \tag{8b}$$

$$u=U_\infty, \quad T=T_\infty \quad \text{at } \theta=0 \tag{8c}$$

where $h=C_pT$ has been introduced, and C ($\equiv\rho\mu/\rho_\infty\mu_\infty$) is the Chapman-Rubesin parameter.

It is now assumed that C can be represented by some constant value across the boundary layer. This being the case, Eqs. (6-8) are equivalent to the equations governing a corresponding incompressible flow with $\nu=C\nu_\infty$. The solutions of the equations for $u(Y,\theta)$ and $T(Y,\theta)$ as given by Chung and Thomas[6] for incompressible flow may thus be adapted. The velocity profile in the transformed plane is

$$u/u_\infty=\text{erf}[(Y/2)\ (C\nu_\infty\theta)^{1/2}] \tag{9}$$

The temperature profile may likewise be inferred. The velocity and temperature profiles thus obtained are valid only for a single eddy during its residence time on the surface. To calculate the mean profiles, we average over all eddies with different contact times, thus

$$\bar{u}=\int_0^\infty u\phi(\theta)\,\mathrm{d}\theta \tag{10}$$

where $\phi(\theta)$ is the fraction of eddies having contact times between θ and $\theta+\mathrm{d}\theta$. Employing Danckwerts [7] random contact time distribution

$$\phi(\theta)=1/\tau \exp[-\theta/\tau] \tag{11}$$

where τ is the mean residence time results in

$$\bar{u}/U_\infty=1-\exp[-Y/(C\nu_\infty\tau)^{1/2}] \tag{12}$$

Employing this result with Eq. (5), the time average wall shear may be calculated

$$\bar{\sigma}_0=\mu_w\frac{\partial\bar{u}}{\partial y}\Big|_0=\mu_w\frac{\partial\bar{u}}{\partial Y}\Big|_0\frac{\mathrm{d}y}{\mathrm{d}Y}=\frac{\rho_w\mu_w u_\infty}{\rho_\infty(C\nu_\infty\tau)^{1/2}} \tag{13}$$

thus we may relate τ to the skin friction coefficient by

$$U^*(\tau/C\nu_\infty)^{1/2}=(2/C_f)^{1/2} \tag{14}$$

where

$$U^*=(\bar{\sigma}_0/\rho_\infty)^{1/2} \tag{15}$$

This result is identical to the result for incompressible flow[1] provided that $\nu=C\nu_\infty$ is used.

Consider now the wall heat flux

$$\bar{q}_0=-k_w\frac{\partial\bar{T}}{\partial y}\Big|_0=-\frac{\rho_w k_w}{\rho_\infty}\frac{\partial\bar{T}}{\partial Y}\Big|_0 \tag{16}$$

The Stanton number may thus be written

$$St=\frac{\bar{q}_0}{\rho_\infty U_\infty C_p\Delta T}=-\frac{C\nu_\infty(\partial\bar{T}/\partial Y)|_0}{Pr\ U_\infty\ \Delta T} \tag{17}$$

The term of the right hand side of Eq. (17) is the identical result of incompressible flow with $\nu=C\nu_\infty$, $Pr=Pr_\infty$. Now the mean temperature profile is a function of τ, thus $St=St(\tau)$ and from Eq. (14) $C_f=C_f(\tau)$. τ is unknown at this point; however τ may be eliminated between Eqs. (15) and (17), resulting in a relationship between St and C_f. Now we have seen that St, C_f, and τ are in fact governed by equivalent incompressible relations with viscosity given by $\nu=C\nu_\infty$. It thus follows that the resulting relationship between St and C_f can be adopted from the work of Chung and Thomas.[6] In addition, the predictions of recovery factor may be written down

Stanton Number‡

$$\frac{\bar{q}_0}{\rho_\infty U_\infty C_p(T_w-T_\infty)}=(Pr)^{-1/2}(1-G)\ (C_f/2) \tag{18}$$

Stanton Number‡

$$\frac{\bar{q}_0}{\rho_\infty U_\infty C_p(T_w-T_{aw})}=(Pr)^{-1/2}(C_f/2) \tag{19}$$

Recovery Factor:

$$R=\frac{4}{\pi}\left[\frac{Pr}{2-Pr}\right]^{1/2}\tan^{-1}\left[\frac{2-Pr}{Pr}\right]^{1/2} \quad Pr<2 \tag{20a}$$

$$=\frac{4}{\pi} \quad Pr=2 \tag{20b}$$

$$=\frac{2}{\pi}\left[\frac{Pr}{2-Pr}\right]^{1/2}\ln\left[\frac{1+(1-2/Pr)^{1/2}}{1-(1-2/Pr)^{1/2}}\right] \quad Pr>2 \tag{20c}$$

where

$$G=\frac{2A\tan^{-1}(2-Pr/Pr)^{1/2}}{[Pr(2-Pr)]^{1/2}} \quad Pr<2 \tag{21a}$$

$$=A \quad Pr=2 \tag{21b}$$

$$=\frac{A\ln\left[\frac{1+(1-2/Pr)^{1/2}}{1-(1-2/Pr)^{1/2}}\right]}{[Pr(Pr-2)]^{1/2}} \quad Pr>2 \tag{21c}$$

$$A=Pr\ U_\infty^2/\pi C_p(T_w-T_\infty) \tag{22}$$

Discussion

Examination of Eqs. (18-22) yields the interesting result that predictions of Reynolds' analogy factor and recovery factor depend only on the Prandtl number, independent of the specific value of the Chapman-Rubesin parameter. A similar result could have been obtained by assuming $C=1$ at the outset; however the generality of the result would not have been obvious. The only restrictions on the analysis, in addition to the assumptions employed in arriving at Eqs. (1-4) is that the value of C must be constant across the boundary layer. Although this is seldom strictly true in practice, White[12] points out that for a flat plate boundary layer at a Mach number of 5, C only varies from 0.6 at the wall to 1.0 in the freestream. The assumption of some constant (average) value for C should not be greatly in error for gas flows at moderate Mach numbers. The excellent agreement obtained by Thomas and Chung[5] and Chung and Thomas[6] between the incompressible flow version of the present theory and compressible flow data is thus substantiated.

It is possible to invert the transformation of Eq. (5) to predict velocity and temperature profiles in the wall region; however since

$$Y=\int_0^y(\rho/\rho_\infty)\,\mathrm{d}\xi=\int_0^y(T_\infty/T)\,\mathrm{d}\xi$$

while

$$T/T_\infty=f(Y,\tau)$$

an implicit equation for T/T_∞ as a function of y,τ will result and no convenient analytic form can be found.

‡Note the two different definitions of ΔT used in Eqs. (18) and (19).

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Errata

Aeroelastic Stability of Periodic Systems with Application to Rotor Blade Flutter

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EQUATION (1) should read:

$$u=-w_e(\beta_p+\beta_D)-(x_0/2)\ (\beta_p+\beta_D)^2+v_e\beta_D\theta$$

$$-\tfrac{1}{2}\int_0^{x_0}\left[\left(\frac{\partial v_e}{\partial x_1}\right)^2+\left(\frac{\partial w_e}{\partial x_1}\right)^2\right]dx_1$$

The legend on Fig. 9 should also include $\theta=0.15$. The superscript on the middle term of the product on the right-hand side of Eq. (20) should be a subscript. The round brackets on the right-hand side of Eqs. (22-24) should contain only the numerator; i.e., $(\Delta_i \underset{\sim}{C}_i)^n/n!$, etc.

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